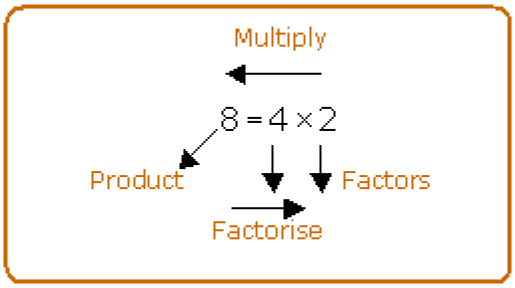
Polynomials

Introduction

An algebraic expression of the form $a_0 + a_1x + a_2x^2 + \ldots. + a_nx^n$ where $a_0, a_1, a_2, \ldots. a_n$ are real numbers, n is a positive integer is called a polynomial in x.
You are familiar with factors and products in the case of numbers.
For example, 8 is the product of 4 and 2. 4 and 2 are called the factors of 8.



Similarly, the algebraic expression a. b. c = abc can be written as 1.a.b.c or 1.ab.c or 1.bc. a or 1.ac. b or 1.abc.
1, a, b, c, ab, bc, ac, abc are all factors of a.b.c and a.b.c is a product.
Factorization is expressing a given expression or number as a product of its factors.

Factorization of Polynomials

You know that any polynomial of the form p(a) can also be written as

$$p(a) = g(a) \times h(a) + R(a) \longrightarrow \text{Dividend} = \text{Quotient} \times \text{Divisor} + \text{Remainder}$$

If the remainder is zero, then p(a) = g(a) x h(a). That is, the polynomial p(a) is a product of two other polynomials g(a) and h(a). For example, $3a + 6a^2 = 3a \times (1 + 2a)$.

It may be possible to express a polynomial as the product of two or more polynomials, in more than one form.

Study the polynomial $3a + 6a^2 = 3a \times (1 + 2a)$.

This can also be factorised as $3a + 6a^2 = 6a \times (\frac{1}{2} + a)$.

Methods of Factorizing Polynomials

There are various methods of factorizing a polynomial. They are,
1. Factorization by dividing the expression by the HCF of the terms of the given expression.
2. Factorization by grouping the terms of the expression.
3. Factorization using identities.

Factorization by Dividing the Expression by the HCF of the Terms of the Given Expression
HCF of a polynomial is the largest monomial, which is a factor of each term of the polynomial. We can factorize a polynomial by finding the Highest Common Factor (HCF) of the terms of the expression and then dividing each term by its HCF. HCF and the quotient obtained are the factors of the given expression.

Steps for Factorization
- Identify the HCF of the terms of the given expression.
- Divide each term of the given expression by the HCF and find the quotient.
- Write the given expression as a product of HCF and quotient.

Factorization by Grouping the Terms of the Expression
In many situations, we come across polynomials, which may not have common factors among its terms. In such cases, we group the terms of the expression in such a way that there are common factors among the terms of the groups so formed.

Steps for Factorization by Grouping
- Rearrange the terms if necessary.
- Group the given expression in such a way that each group has its common factor.
- Identify the HCF of each group.
- Identify the other factor.
- Write the expression as a product of the common factor and the other factor.

Factorization using Identities
Recall the following identities for finding the products:

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$(a+b)(a-b) = a^2 - b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

Observe that the LHS in the identities are all factors and the RHS are their products. Thus, we can write the factors as follows:

Factors of $a^2 - 2ab + b^2$ are $(a-b)$ and $(a-b)$

Factors of $a^2 + 2ab + b^2$ are $(a+b)$ and $(a+b)$

Factors of $a^2 - b^2$ are $(a+b)$ and $(a-b)$

Factors of $a^3 + 3a^2b + 3ab^2 + b^3$ are $(a+b)$, $(a+b)$ and $(a+b)$

Factors of $a^3 - 3a^2b + 3ab^2 - b^3$ are $(a - b)$, $(a - b)$ and $(a - b)$

Factors of $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$ are $(a + b + c)$ and $(a + b + c)$

From the above identities we observe that a given expression, which is in the form of an identity can be written in terms of its factors.

Steps for factorization using Identities

- Recognize the appropriate identity.
- Rewrite the given expression in the form of the identity.
- Write the factors of the given expression using the identity.

$$a^3 \pm b^3 \pm 3ab(a \pm b) = (a \pm b)^3$$

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

$$x^3 \pm y^3 = (x \pm y)(x^2 \pm xy + y^2)$$

Factorization of Trinomials of the form x^2+bx+c

Trinomials are expressions with three terms.

For example, $x^2 + 14x + 49$ is a trinomial. There is no single method by which all trinomials can be factorized. We need to study the pattern in trinomials and choose the appropriate method to factorize the given trinomial.

Factorizing a Trinomial by Splitting the Middle Term

The product of two binomials of the type $(x + a)$ and $(x + b)$ is

$(x + a) \times (x + b) = x^2 + x(a + b) + ab$ [a trinomial].

In these examples, study the relation between the middle and the last terms.

Middle Term	Observe that the numerical	Factors of last term
a) $12x = 9x + 3x$	coefficient of the middle term is the	$27 = 9 \times 3$
b) $12s = 8s + 4s$	sum of the factors of last term.	$32 = 8 \times 4$
c) $12m = 7m + 5m$	∴ To factorise a trinomial expression of the type $x^2 + cx + d$ look for this relation.	$35 = 7 \times 5$

Therefore, to factorize expressions of the type $(x^2 + cx + d)$, we have to find two factors which satisfy the above condition. That is, we need to split the middle term so that the product of the factors is equal to the last term.

Steps to Factorize a Trinomial of the form $x^2 + bx + c$ where b and c are Integers:

- Find all pairs of factors whose product is the last term of the trinomial.
- From the pairs of factors from step 1, choose a pair of factors whose sum is the coefficient of the middle term of the trinomial.
- Split the middle term using the pair of factors from step 2 and rewrite the trinomial.
- Group the terms from step 3 and factorize.
- Verify the solution.

Factoring a trinomial of the type $ax + bx^2 + c$ $(a \neq 1)$ by splitting the middle term

To factorize expressions of the type $x^2 + bx + c$, you will find two numbers a and b such that their sum is equal to the coefficient of the middle term and their product is equal to the last term(constant).

Steps for factoring $ax^2 + bx + c$ $(a \neq 1)$ by grouping

- Find the product (ac), of the coefficient of x^2 and the last term.
- List the factor pairs of ac.
- Choose a factor pair whose sum is the coefficient of the middle term.
- Rewrite the polynomial by splitting the middle term.
- Regroup and factorize.

Remainder Theorem

If f(x) is a polynomial in x and is divided by x-a; the remainder is the value of f(x) at x = a i.e., Remainder = f(a).

Proof:

Let p(x) be a polynomial divided by (x-a).

Let q(x) be the quotient and R be the remainder.

By division algorithm,

Dividend = (Divisor x quotient) + Remainder

p(x) = q(x) . (x-a) + R

Substitute x = a,

p(a) = q(a) (a-a) + R

p(a) = R (a - a = 0, 0 - q (a) = 0)

Hence Remainder = p(a).

Steps for Factorization using Remainder Theorem

- By trial and error method, find the factor of the constant for which the given expression becomes equal to zero.
- Divide the expression by the factor that is determined in step 1.
- Factorize the quotient. If the quotient is a trinomial, factorize it further.
- If the expression is a 4^{th} degree expression, the first step will be to reduce this to a trinomial and then factorize this trinomial further.

Factor Theorem

Statement:

If p(x), a polynomial in x is divided by x-a and the remainder = p (a) is zero, then (x-a) is a factor of p(x).

Proof:
When p(x) is divided by x-a,
R = p(a) (by remainder theorem)
p(x) = (x-a).q(x)+p(a)
(Dividend = Divisor x quotient + Remainder Division Algorithm)
But p(a) = 0 is given.
Hence p(x) = (x-a).q(x)
⇒ (x-a) is a factor of p(x).
Conversely if x-a is a factor of p(x) then p(a) = 0.
p(x) = (x-a).q(x) + R
If (x-a) is a factor, then the remainder should be zero (x - a divides p(x) exactly)
R = 0
By remainder theorem, R = p(a)
⇒p(a) = 0